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# Canonical Forms of Multidimensional Steady Inviscid Flows

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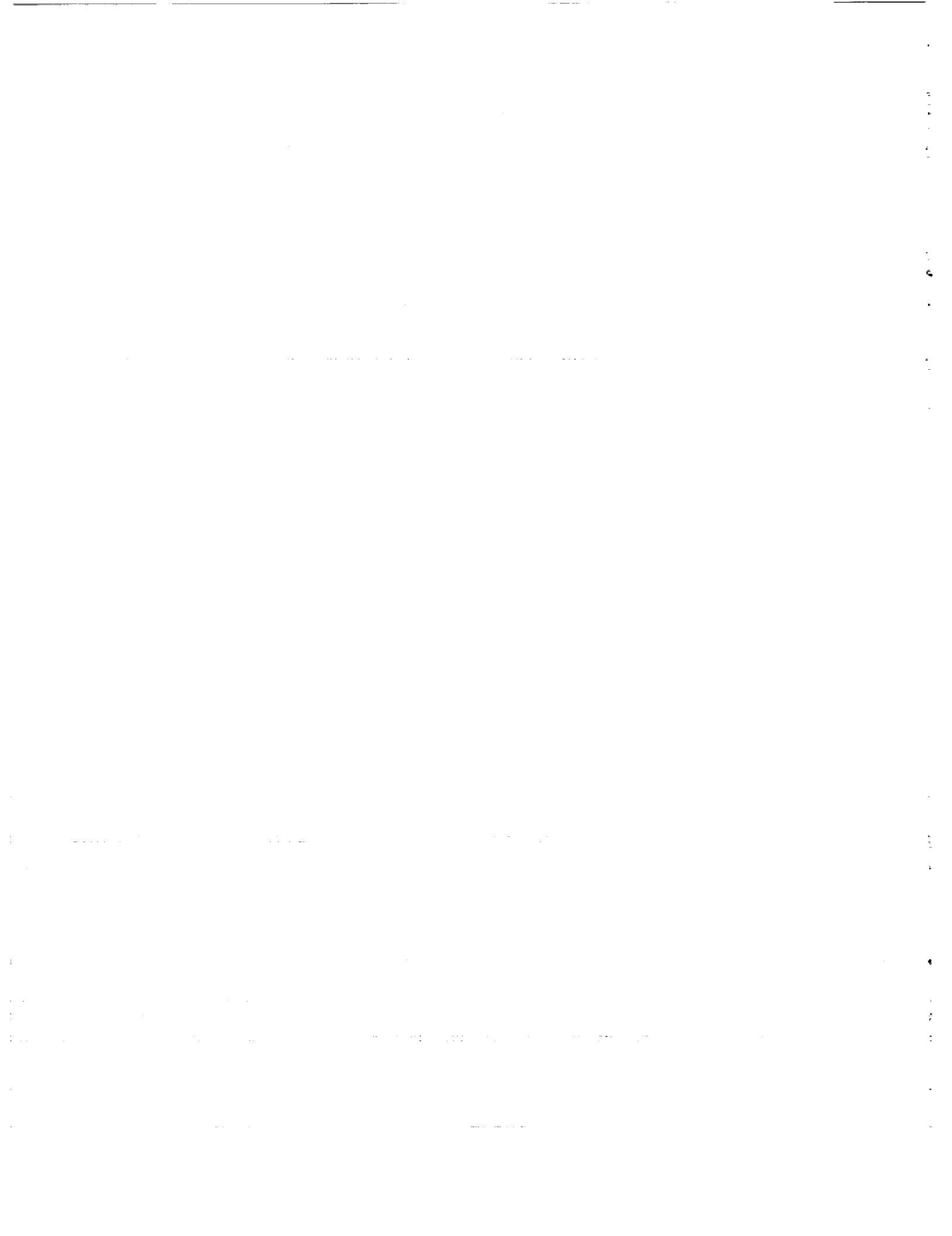
## Abstract

Canonical forms and canonical variables for inviscid flow problems are derived. In these forms the components of the system governed by different types of operators (elliptic and hyperbolic) are separated. Both the incompressible and compressible cases are analyzed and their similarities and differences are discussed. The canonical forms obtained are block upper triangular operator form in which the elliptic and non-elliptic parts reside in different blocks. The full nonlinear equations are treated without using any linearization process. This form enables a better analysis of the equations as well as better numerical treatment. These forms are the analog of the decomposition of the one dimensional Euler equations into characteristic directions and Riemann invariants.

**Key Words:** canonical forms, inviscid flow, Euler equations

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# 1 Introduction

In the past decade a substantial effort has been invested in numerical solutions of the Euler equations in one dimension. These were based on characteristic decomposition of the flow, i.e., a decomposition into three subsystems each of which correspond to a simple propagation along an appropriate line in space-time. Efficient discretization schemes were based on this representation. We refer to [2] for extensive reference on the subject. This approach is usually extended to multidimensional problems by splitting it into a sequence of one-dimensional problems.

The above approach is limited and does not capture the real structure of the solution for multidimensional flow problems. Moreover, the analysis is usually based on the time-dependent problem although the case of interest is the steady-state problem. Consequently, an important feature of the system such as its mixed nature has not been given proper attention. This paper is concerned with multidimensional steady-state inviscid flow problems.

An analysis which is analogous to the decomposition of the one dimensional Euler equations into characteristic directions and the Riemann invariants, is presented for multidimensional steady-state inviscid flow problems. This methodology can be applied to more general partial differential equations, such as the steady state and time dependent viscous flow equations. This analysis of the equations break up the full problem into its fundamental (irreducible) subproblems and also describe the interaction between them. It leads to a better understanding of the equations and to new type of schemes that better represent the physical behavior.

The analysis is presented for incompressible and compressible Euler equations both both in two and three dimensions. It shows in a clear way the existence of quantities that propagate along streamlines, and other quantities whose behavior is directionally unbiased for subsonic compressible flows or incompressible flows. Supersonic flows, on the other hand, both in two and three dimensions are hyperbolic with respect to the stream direction. A change of variables is introduced in order to bring the system into its canonical form, and the resulting new variables are called canonical variables. The canonical forms are block upper triangular form, where the diagonal blocks consists of the basic components of the systems and off diagonal ones represent some interaction between the different subsystems.

For two dimensional incompressible Euler equations the canonical variables are velocities and total pressure  $p + q^2/2$  and in three dimensions they are velocities, total pressure and normalized helicity, namely,  $(\omega \cdot \mathbf{V})/(\mathbf{V} \cdot \mathbf{V})$ . Canonical variables other than the velocities propagate along streamlines and form the non-elliptic part of the system. The velocity components satisfy an elliptic system with a forcing depending on the hyperbolic parts.

For the the two dimensional compressible Euler equations the canonical variables are velocities and total enthalpy, and in three dimensions a normalized helicity, namely,  $(\omega \cdot \rho \mathbf{V})/(\rho \mathbf{V} \cdot \rho \mathbf{V})$  is the additional canonical variable. Quantities other than the velocities propagate along streamlines. The velocities are governed by an elliptic system for subsonic regimes and by a hyperbolic system in supersonic regimes. The structure of the canonical forms shows in a clear way the relation between the full problems (incompressible and compressible Euler equations) and simplified models that have been studied in the past, such as (div, curl) systems in two and three dimensions, and their compressible versions. Canonical boundary conditions are discussed in this framework. These are the analog of characteristic boundary conditions for one dimensional compressible flows.

## 2 On Canonical Forms for Systems of Partial Differential Equations

Differential equations of constant coefficient are reduced by the Fourier transform to algebraic equations in terms of symbols which are in general matrix polynomials in several variables. Let  $P(D)$  be a differential operator of the form

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha \quad (2.1)$$

where  $D = (D_1, \dots, D_d)$ ,  $\alpha = (\alpha_1, \dots, \alpha_d)$  and  $D^\alpha = D_1^{\alpha_1} \dots D_d^{\alpha_d}$  is the usual multi-index notation. The  $a_\alpha$  are  $l \times l$  real valued matrices. We want to study solutions to the problem

$$P(D)u(x) = f(x) \quad x \in \Omega \quad (2.2)$$

$$B(D)u(x) = g(x) \quad x \in \partial\Omega \quad (2.3)$$

where  $\Omega$  is a domain in  $\mathbb{R}^d$  and  $B(D)$  is an appropriate boundary condition to be specified later.

Fourier analysis in full and half space are carried out in order to analyze the interior and boundary properties of the solutions, respectively. The symbol associated with  $P(D)$  is the polynomial

$$\hat{P}(x, i\xi) = \sum_{|\alpha| \leq m} a_\alpha (i\xi)^\alpha \quad (2.4)$$

where  $\xi = (\xi_1, \dots, \xi_d)$  and  $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_d^{\alpha_d}$  and its use transforms the constant coefficient differential equation into the algebraic equation

$$\hat{P}(i\xi)\hat{u}(\xi) = \hat{f}(\xi). \quad (2.5)$$

For elliptic systems where the matrix polynomial satisfies

$$|\det \hat{P}(i\xi)| \geq C|\xi|^{2m} \quad (2.6)$$

$$(2.7)$$

extensive research has been done with regard to smoothness properties of the solution including the effect of boundary conditions. The case of hyperbolic systems of differential equations, i.e., when the principal part of  $\det \hat{P}(i\xi)$  has real roots for  $\xi \neq 0$ , has been studied in the context of time dependent problems.

For the applications discussed in this paper one is interested in analyzing steady-state problems which are mixed elliptic-hyperbolic. That is, part of the system is governed by hyperbolic equations (with respect to some direction in space) and the rest of the system by an elliptic subsystem, and some interaction exists between the two subsystems.

The first step in the analysis of a mixed system is the identification of the different subsystems, i.e., elliptic versus hyperbolic. Next comes the identification of the variables corresponding to

different type of behaviors, followed by a reduction of the system to block triangular form. The diagonal blocks are the basic building blocks of the system.

We discuss two types of reduction to canonical forms. In the first case the new set of variables are linear combination of the old ones, via constant matrices. In the other case a linear combination via polynomials is used. Thus, the new variables in the second method involve linear combinations of the old ones and their derivatives.

### C-Reducibility

Assume that there exist invertible matrices  $S, T$ , (independent of  $\xi$ ) such that  $S\hat{P}(\xi)T$  is of block upper triangular form, i.e.,

$$S\hat{P}(\xi)T = \begin{pmatrix} \hat{P}_{11}(\xi) & \hat{P}_{12}(\xi) & \dots & \hat{P}_{1k}(\xi) \\ & \hat{P}_{22}(\xi) & & \\ & & \ddots & \\ & & & \hat{P}_{kk}(\xi) \end{pmatrix} \equiv \Lambda \quad (2.8)$$

where the subsystems  $\hat{P}_{jj}, j = 1, \dots, k$  are irreducible.

Taking the determinant of both sides, and assuming that  $\det(S)\det(T) = 1$ , we obtain

$$\det \hat{P}(\xi) = \det \hat{P}_{11}(\xi) \det \hat{P}_{22}(\xi) \dots \det \hat{P}_{kk}(\xi) \quad (2.9)$$

The blocks on the diagonal, correspond therefore, to the primitive building blocks of the system, and the off diagonal ones represent the interaction of the different basic blocks.

**Definition 1** A matrix polynomial  $\hat{P}(\xi)$  will be called C-reducible if there exist invertible matrices  $S, T$  independent of  $\xi$  such that  $\Lambda(\xi) = S\hat{P}(\xi)T$  is of block triangular form.

We will refer to this form of the matrix polynomial as the canonical form. It induces a canonical form for the differential operator via the Fourier transform.

Thus, we have the following theorem,

**Theorem:** A necessary condition for an  $l \times l$  matrix polynomial  $\hat{P}(i\xi)$  to be C-reducible to a block triangular form is that its determinant admits a factorization into lower order polynomials.

We give some examples to illustrate some important points with respect to the canonical form.

**Example I:** The symbol associated with the Cauchy-Riemann equations is

$$\begin{pmatrix} i\xi_1 & i\xi_2 \\ i\xi_2 & -i\xi_1 \end{pmatrix} \quad (2.10)$$

Its determinant  $-\xi_1^2 - \xi_2^2$  is irreducible, therefore, this matrix is not C-reducible.

The factorization of the determinant into polynomials of smaller degree does not guarantee that the corresponding matrices are C-reducible.

**Example II:** The matrix

$$\begin{pmatrix} i\xi_1 & i\xi_2 & i\xi_3 \\ i\xi_2 & i\xi_1 & 0 \\ i\xi_3 & 0 & i\xi_1 \end{pmatrix} \quad (2.11)$$

whose determinant

$$(i\xi_1)(-\xi_1^2 + \xi_2^2 + \xi_3^2) \quad (2.12)$$

is a product of two lower order polynomials, is not C-reducible. This can be seen by observing that the eigenvector corresponding to the eigenvalue  $i\xi_1$  depends on  $\xi$ .

### P-Reducibility

For numerical applications one may allow a more general transformation in reducing a system to block triangular form as suggested by the following definition.

**Definition 2:** A matrix polynomial  $\hat{P}(\xi)$  is said to be P-reducible if there exist invertible matrix polynomials  $S(\xi), T^{-1}(\xi)$  such that  $S(\xi)\hat{P}(\xi) = \Lambda(\xi)T^{-1}(\xi)$ , where  $\Lambda(\xi)$  is a matrix polynomial of block triangular form.

The matrix given in Example is P-reducible. This can be seen from the identity

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & i\xi_2 & i\xi_3 \\ 0 & i\xi_3 & -i\xi_2 \end{pmatrix} \begin{pmatrix} i\xi_1 & i\xi_2 & i\xi_3 \\ i\xi_2 & i\xi_1 & 0 \\ i\xi_3 & 0 & i\xi_1 \end{pmatrix} = \begin{pmatrix} i\xi_1 & 1 & 0 \\ |\xi|^2 & i\xi_1 & 0 \\ 0 & 0 & i\xi_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & i\xi_2 & i\xi_3 \\ 0 & i\xi_3 & -i\xi_2 \end{pmatrix} \quad (2.13)$$

Thus, using

$$S(\xi) = T^{-1}(\xi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i\xi_2 & i\xi_3 \\ 0 & i\xi_3 & -i\xi_2 \end{pmatrix} \quad (2.14)$$

we get a reduction to a block upper triangular form given by

$$\Lambda = \begin{pmatrix} i\xi_1 & 1 & 0 \\ |\xi|^2 & i\xi_1 & 0 \\ 0 & 0 & i\xi_1 \end{pmatrix} \quad (2.15)$$

By using the operators corresponding to  $S, T$  for the differential problem one may end up with other variables than the desired ones. This reduction is still important for analyzing the equation but the original variables may need to be solved for as well. One can define an extended system in which the canonical form is combined with the transformation of variable and get a block triangular form as

$$\begin{pmatrix} T^{-1} & -I \\ 0 & \Lambda \end{pmatrix} \quad (2.16)$$



This form which we call the extended canonical form may be required only when the matrices  $S, T^{-1}$  depend on  $\xi$ . In the other case the transformation from the canonical variables to the original ones is done by an algebraic mapping. In the applications discussed in the next section the extended canonical form will be found useful for three dimensional problem.

Assuming that  $S, T$  transform the matrix polynomial  $\hat{P}(\xi)$  into block triangular form, we obtain a similar reduction for the differential equation, namely,

$$(SP(D)T)(T^{-1}u) = Sf \quad (2.17)$$

that is, in terms of a new set of variables  $T^{-1}u$  we get

$$\Lambda(D)v = Sf \quad (2.18)$$

The boundary conditions  $B(D)u = g$  are transformed into

$$\sum_{i,j=1,k} B_{ij}(D)v_j = (Qg)_i \quad i = 1, \dots, k \quad (2.19)$$

and we assume that for each  $j = 1, \dots, k$  the operator  $P_{jj}(D)$  with the boundary condition  $B_{jj}(D)$  form a well posed problem.

The simplest set of boundary conditions are those for which the boundary operator in the transformed variables is diagonal, and each corresponding subproblem is well posed. We refer to a set of such boundary conditions as canonical.

All the arguments made above applies to linear constant coefficient problems. The variable coefficient case and the nonlinear case may have triangular forms for the principal part only.

It should be mentioned that the use of  $S, T^{-1}$  which are not polynomials can reduce  $\hat{P}(\xi)$  to upper triangular form. However, the elements of this form are not polynomials, therefore, do not correspond to differential operators. While such a representation is still useful in analysis of well posedness of the boundary value problem, it is not as useful in developing numerical methods for the problem. The representations discussed in this paper involve transformations that correspond to differential operators and can be used in numerical implementation of certain iterative techniques that are based on these forms. These issues will be discussed in a separate paper.

### 3 Canonical Forms for Incompressible Flow

In the applications we present in this and the next section, although nonlinear, a full reduction to triangular form is presented, without involving any linearization.

The steady state incompressible inviscid equations in nonconservative formulation [1] are given by

$$\begin{aligned} \operatorname{div} \mathbf{V} &= 0 \\ (\mathbf{V} \cdot \nabla) \mathbf{V} + \nabla p &= 0 \end{aligned} \quad (3.1)$$

which can be written in operator form as

$$\begin{pmatrix} \text{div} & 0 \\ \mathbf{Q} & \text{grad} \end{pmatrix} \begin{pmatrix} \mathbf{V} \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.2)$$

where  $\mathbf{Q}$  is a diagonal matrix operator of dimension  $d \times d$  whose diagonal entries are the scalar convection operator  $Q = \mathbf{V} \cdot \nabla$ .

Analyzing the determinant of this system reveals its structure which is necessary for analyzing possible boundary conditions and discretization issues. Denoting the matrix operator above by  $L_{inc}$ , and freezing the coefficients at some constant flow,  $\mathbf{V}_0$ , one obtains

$$\det \hat{L}_{inc}(\xi) = -|\xi|^2 (\mathbf{V}_0 \cdot \xi) \quad (3.3)$$

where,  $\xi = (\xi_1, \dots, \xi_d)$ ,  $|\xi|^2 = \xi_1^2 + \dots + \xi_d^2$ .

The determinant which is already factored into irreducible factors reveals important properties of the system of the incompressible Euler equations. While the operator  $\Delta$  is elliptic, the operator  $Q$  is hyperbolic with respect to the stream direction. This system is therefore of a mixed type. Some components of the system display hyperbolic behavior, while the rest show elliptic behavior. The power  $d - 1$  in the factorization of the determinant suggests that there are  $d - 1$  quantities that are governed by hyperbolic subsystems. This forms a subsystem of the full equation which is weakly coupled to the rest.

We proceed by identifying the hyperbolic components of this system. Using the relation

$$(\mathbf{V} \cdot \nabla) \mathbf{V} = \frac{1}{2} \nabla (\mathbf{V} \cdot \mathbf{V}) - \mathbf{V} \times \text{curl} \mathbf{V} \quad (3.4)$$

and defining

$$P = p + \frac{1}{2} q^2 \quad q^2 = \mathbf{V} \cdot \mathbf{V} \quad (3.5)$$

one obtain Crocco's form of the equations of motion, namely,

$$\begin{aligned} \text{div}(\mathbf{V}) &= 0 \\ -\mathbf{V} \times \text{curl} \mathbf{V} + \nabla P &= 0 \end{aligned} \quad (3.6)$$

Let  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  be an orthonormal basis such that  $\mathbf{e}_1$  is in the direction of  $\mathbf{V}$ , and  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  form a right hand system. i.e.,  $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ ,  $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$ ,  $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$

Taking the inner product of the momentum equations with  $\mathbf{V}$  and using  $\mathbf{V} \cdot (\mathbf{V} \times \text{curl} \mathbf{V}) = (\mathbf{V} \times \mathbf{V}) \cdot \text{curl} \mathbf{V} = 0$ , we obtain

$$\mathbf{V} \cdot \nabla P = 0 \quad (3.7)$$

This equation implies that the total pressure is constant along streamlines. Thus, the total pressure  $P$  is one of the canonical variables we are looking for, and it corresponds to a factor  $\mathbf{V} \cdot \nabla$  in the factorization of the system. For three dimensional flows there exists another quantity that

is governed by a hyperbolic operator. From physical insight, a quantity that exists in three-dimensional and not in two dimensional flows may have to do with vorticity. The vorticity  $\omega$  defined by

$$\omega = \text{curl} \mathbf{V} \quad (3.8)$$

will be shown to play an important role in the decomposition of the system.

It can be easily verified that the vorticity component in the direction of the velocity cannot be determined from the momentum equations. However, we can obtain the following

$$-\mathbf{e}_j \cdot (\mathbf{V} \times \omega) + \mathbf{e}_j \cdot \nabla P = 0, \quad j = 2, 3 \quad (3.9)$$

which gives

$$-(\mathbf{e}_j \times \mathbf{V}) \cdot \omega + \mathbf{e}_j \cdot \nabla P = 0, \quad j = 2, 3 \quad (3.10)$$

Using the definition of  $\mathbf{e}_j$  we get the following equations

$$\begin{aligned} -q(\mathbf{e}_2 \cdot \omega) + \mathbf{e}_3 \cdot \nabla P &= 0 \\ q(\mathbf{e}_3 \cdot \omega) + \mathbf{e}_2 \cdot \nabla P &= 0 \end{aligned} \quad (3.11)$$

which can be interpreted as equations that determine two components of the vorticity vector.

The other component of the vorticity is obtained from

$$\text{div} \omega = 0 \quad (3.12)$$

which is a compatibility condition ( $\text{div} \text{curl} = 0$ ). Decomposing  $\omega$  as

$$\begin{aligned} \omega &= \beta \mathbf{V} + \omega^\perp \\ \omega^\perp \cdot \mathbf{e}_1 &= 0 \end{aligned} \quad (3.13)$$

and substituting (3.13) in (3.12) and using the continuity equation one obtain an equation for  $\beta$ , namely,

$$\mathbf{V} \cdot \nabla \beta + \text{div} \omega^\perp = 0 \quad (3.14)$$

Therefore,  $\beta$  is also governed by a hyperbolic equation. Note that while  $P$  is constant along streamlines,  $\beta$  admits a more complicated structure depending on the behavior of  $P$  in cross-stream directions.

Summarizing, we get the following

$$\begin{aligned} \text{div} \mathbf{V} &= 0 \\ \text{curl} \mathbf{V} - \omega &= 0 \\ \mathbf{V} \cdot \nabla \beta + \text{div}(\omega^\perp) &= 0 \\ \omega^\perp &= (\mathbf{e}_2 \cdot \omega^\perp) \mathbf{e}_2 + (\mathbf{e}_3 \cdot \omega^\perp) \mathbf{e}_3 \\ -q(\mathbf{e}_2 \cdot \omega^\perp) + \mathbf{e}_3 \cdot \nabla P &= 0 \\ q(\mathbf{e}_3 \cdot \omega^\perp) + \mathbf{e}_2 \cdot \nabla P &= 0 \\ q\mathbf{e}_1 \cdot \nabla P &= 0 \end{aligned} \quad (3.15)$$

which can be written in a matrix form as

$$\left( \begin{array}{c|ccc|c} \text{div} & & & & \\ \hline -\text{curl} & q\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \\ \hline & \mathbf{V} \cdot \nabla & \text{dive2} & \text{dive3} & \\ & 0 & -q & 0 & \mathbf{e}_3 \cdot \nabla \\ & 0 & 0 & q & \mathbf{e}_2 \cdot \nabla \\ & & & & \mathbf{V} \cdot \nabla \end{array} \right) \begin{pmatrix} \mathbf{V} \\ \beta \\ \omega_2 \\ \omega_3 \\ P \end{pmatrix} = 0 \quad (3.16)$$

Observe that although the upper block of the system which consists of four equations ( $\text{div}, \text{curl}$ ) is overdetermined, its solution is guaranteed by the presence of the  $\beta$  equation, which states that  $\text{div}\omega = 0$ . Another way of looking at it is to eliminate the  $\text{div}\omega = 0$  equation and to regard the ( $\text{div}, \text{curl}$ ) system as an equation for both  $V$  and  $\beta$ . In this way we obtain a 4 by 4 system for four unknown quantities. This fixes a well known difficulty of solving ( $\text{div}, \text{curl}$ ) in three space dimensions.

For two-dimensional flows  $\beta = 0$  and the equations take a simpler form. If  $\mathbf{e}_1, \mathbf{e}_2$  lie in the plane defined by the flow then  $\omega^\perp \cdot \mathbf{e}_2 = 0$ . Thus,  $\omega$  can be represented by a scalar quantity which we denote by  $\omega$ . The canonical form of the two dimensional incompressible Euler equations reduces to

$$\left( \begin{array}{cc|cc} D_x & D_y & 0 & 0 \\ D_y & -D_x & -1 & 0 \\ \hline 0 & 0 & -q & \frac{1}{q}D_0 \\ 0 & 0 & 0 & Q \end{array} \right) \begin{pmatrix} u \\ v \\ \omega \\ P \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.17)$$

where

$$D_0 = vD_x - uD_y \quad (3.18)$$

These representations of the incompressible inviscid equations reveal the structure of solutions. That is, the total pressure  $P$  and the normalized helicity (with respect to velocity) evolve along streamlines, and determine the vorticity vector field. The velocity components satisfy an elliptic system of a ( $\text{div}, \text{curl}$ ) form.

In this representation which we call the canonical form of the incompressible Euler equations the elliptic part has been separated from the hyperbolic part. The variables  $(V, \beta, P)$  will be referred to as the canonical variables.

This form of the equations suggest a natural set of boundary conditions, which we refer to as canonical boundary conditions for the corresponding systems. The appearance of the Laplacian operator  $\Delta$  in the factorization of the determinant calls for one boundary condition at every boundary point. The term  $Q^{d-1}$  implies that additional  $d - 1$  boundary conditions are to be prescribed at inflow points of the boundary. From the factorization it is natural to prescribe at inflow the canonical variables  $P, \beta$ , and the extra condition to be imposed at every point is  $\mathbf{V} \cdot \mathbf{n}$  where  $\mathbf{n}$  is an outward normal. Note that for two-dimensional problems  $\beta = 0$  and therefore one boundary condition is omitted.

## 4 Canonical Forms for Compressible Flows

The Euler equations in conservation form [1] in terms of the variables  $(V, p, H)$  are

$$\begin{aligned} \operatorname{div}(\rho \mathbf{V}) &= 0 \\ \operatorname{div}(\rho \mathbf{V} \otimes \mathbf{V} + pI) &= 0 \\ \operatorname{div}(\rho \mathbf{V} H) &= 0 = 0 \end{aligned} \quad (4.1)$$

The analysis of the equations is simplified if one moves to the non-conservative formulation,

$$\begin{pmatrix} Q_c & Q_p & Q_H \\ \rho \mathbf{Q} & \operatorname{grad} & 0 \\ 0 & 0 & \rho Q \end{pmatrix} \begin{pmatrix} \mathbf{V} \\ p \\ H \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{0} \\ 0 \end{pmatrix} \quad (4.2)$$

where

$$\begin{aligned} Q_c &= \rho \operatorname{div} + 2 \frac{\partial \rho}{\partial(q^2)} \mathbf{V} \cdot (\mathbf{V} \cdot \nabla) \\ Q_p &= \frac{\partial p}{\partial p} Q \\ Q_H &= \frac{\partial H}{\partial H} Q \end{aligned} \quad (4.3)$$

Denoting the matrix operator above by  $L_e$ , and freezing the coefficients at  $(\mathbf{V}_0, p_0, H_0)$ , the determinant of the symbol gives

$$\det \hat{L}_e(\xi) = \rho_0^{d+2} (\mathbf{V}_0 \cdot \xi)^d ((\mathbf{V}_0 \cdot \xi)^2 - c_0^2 |\xi|^2) \quad (4.4)$$

where  $c_0$  is the speed of sound and can be evaluated using the relations (where subscripts are omitted),

$$\begin{aligned} s &= p/\rho^\gamma \\ H &= \frac{\gamma}{\gamma-1} s \rho^{\gamma-1} + \frac{1}{2} q^2 \\ \frac{\gamma}{\gamma-1} c^2 &= H - \frac{1}{2} q^2 \\ c^2 &= \gamma p/\rho \end{aligned} \quad (4.5)$$

The factor  $\mathbf{V}_0 \cdot \xi$  appearing in the determinant here is the same one as for the incompressible equation, suggesting that there are  $d$  quantities that propagate along streamlines. The rest of the system is governed by a system equivalent to the well known full potential operator  $Q^2 - c^2 \Delta$  whose character depends on the Mach number  $M$ ,

$$M^2 = q^2/c^2 \quad (4.6)$$

leading to an elliptic equation for  $M < 1$  and hyperbolic (in the stream direction) for  $M > 1$ . As before we use Crocco's form of the equation for analysis purposes.

$$\begin{aligned} Q_c \mathbf{V} &= 0 \\ -\rho \mathbf{V} \times \boldsymbol{\omega} - \rho T \nabla s + \rho \nabla H &= 0 \\ \operatorname{div}(\rho \mathbf{V} H) &= 0. \end{aligned} \quad (4.7)$$

By subtracting  $H$  times the continuity equation from the energy equation one obtains

$$\rho \mathbf{V} \cdot \nabla H = 0 \quad (4.8)$$

Taking the scalar product of the momentum equation with  $\mathbf{V}$  leads to

$$\rho \mathbf{V} \cdot \nabla H - \rho T \mathbf{V} \cdot \nabla s = 0 \quad (4.9)$$

Thus, two of the quantities that propagate along streamlines are the total enthalpy  $H$  and the entropy  $s$ . As the determinant suggest there is one more quantity in three dimensional flow that propagate along streamlines.

To find that quantity we follow a similar path as in the incompressible case by introducing the vorticity and its decomposition as

$$\begin{aligned} \omega &= \text{curl} \mathbf{V} \\ \omega &= \beta \rho \mathbf{V} + \omega^\perp \\ \omega^\perp \cdot \mathbf{V} &= 0 \end{aligned} \quad (4.10)$$

The equation for  $\beta$  is obtained the same way as in the incompressible case, i.e., using  $\text{div} \omega = 0$ , giving

$$\rho \mathbf{V} \cdot \nabla \beta + \text{div} \omega^\perp = 0 \quad (4.11)$$

The quantity  $\omega^\perp$  satisfies

$$\begin{aligned} -\rho q(\mathbf{e}_2 \cdot \omega^\perp) - \rho T \mathbf{e}_3 \cdot \nabla s + \rho \mathbf{e}_3 \cdot \nabla H &= 0 \\ \rho q(\mathbf{e}_3 \cdot \omega^\perp) - \rho T \mathbf{e}_2 \cdot \nabla s + \rho \mathbf{e}_2 \cdot \nabla H &= 0 \end{aligned} \quad (4.12)$$

in complete analogy with the incompressible case.

The continuity equation is simplified by subtracting from it appropriate multiples of the energy and the entropy equations, given  $Q_c \mathbf{V} = 0$ .

Summarizing we get

$$\begin{aligned} Q_c \mathbf{V} &= 0 \\ \text{curl} \mathbf{V} - \omega &= 0 \\ \rho \mathbf{V} \cdot \nabla \beta + \text{div}(\omega^\perp) &= 0 \\ \omega^\perp - (\mathbf{e}_2 \cdot \omega^\perp) \mathbf{e}_2 - (\mathbf{e}_3 \cdot \omega^\perp) \mathbf{e}_3 &= 0 \\ -\rho q(\mathbf{e}_2 \cdot \omega^\perp) - \rho T \mathbf{e}_3 \cdot \nabla s + \rho \mathbf{e}_3 \cdot \nabla H &= 0 \\ \rho q(\mathbf{e}_3 \cdot \omega^\perp) - \rho T \mathbf{e}_2 \cdot \nabla s + \rho \mathbf{e}_2 \cdot \nabla H &= 0 \\ -\rho T \mathbf{V} \cdot \nabla s &= 0 \\ \rho \mathbf{V} \cdot \nabla H &= 0 \end{aligned} \quad (4.13)$$

or in matrix form

$$\left( \begin{array}{c|ccc|cc} Q_c & & & & & \\ \hline -\text{curl} & \rho q \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & & \\ \hline & \rho \mathbf{V} \cdot \nabla & \text{dive}_2 & \text{dive}_3 & 0 & 0 \\ & 0 & -q & 0 & -\rho T \mathbf{e}_3 \cdot \nabla & \rho \mathbf{e}_3 \cdot \nabla \\ & 0 & 0 & q & -\rho T \mathbf{e}_2 \cdot \nabla & \rho \mathbf{e}_2 \cdot \nabla \\ \hline & & & & -\rho T \mathbf{V} \cdot \nabla & 0 \\ & & & & & \mathbf{V} \cdot \nabla \end{array} \right) \begin{pmatrix} \mathbf{V} \\ \beta \\ \omega_2 \\ \omega_3 \\ s \\ H \end{pmatrix} = 0 \quad (4.14)$$

The nature of the subsystem

$$Q_c \mathbf{V} = 0 \quad (4.15)$$

$$\text{curl} \mathbf{V} = \boldsymbol{\omega} \quad (4.16)$$

can be studied by taking the gradient of the first equation and subtracting from it the curl of the second equation. Using the relation

$$\nabla^2 \mathbf{V} = \text{grad div} \mathbf{V} - \text{curl curl} \mathbf{V} \quad (4.17)$$

and

$$\mathbf{V} \cdot \text{grad} \rho = \frac{\partial \rho}{\partial (q^2/2)} \mathbf{V} \cdot (\mathbf{V} \cdot \nabla \mathbf{V}) \quad (4.18)$$

we get

$$\nabla^2 \mathbf{V} + \text{grad} \left( \frac{1}{\rho} \frac{\partial \rho}{\partial (q^2/2)} \mathbf{V} \cdot (\mathbf{V} \cdot \nabla \mathbf{V}) \right) = -\text{curl} \boldsymbol{\omega} \quad (4.19)$$

This equation is elliptic for subsonic flow and hyperbolic with respect to the stream direction for supersonic flows.

In two dimensions  $\beta = 0$  and the canonical form of the Euler equation reduces to

$$\left( \begin{array}{cc|cc} D_1 & D_2 & 0 & 0 \\ -D_y & D_x & -1 & 0 \\ \hline 0 & 0 & -q & -\frac{c^2}{\gamma(\gamma-1)} D_3 \\ 0 & 0 & 0 & T \rho Q \\ 0 & 0 & 0 & 0 \end{array} \right) \begin{pmatrix} u \\ v \\ \omega \\ s \\ H \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (4.20)$$

where

$$\begin{aligned} D_1 &= \rho/c^2((c^2 - u^2)D_x - uvD_y) \\ D_2 &= \rho/c^2((c^2 - v^2)D_y - uvD_x) \\ D_3 &= vD_x - uD_y \end{aligned} \quad (4.21)$$

From the canonical form one can easily see that a natural set of boundary conditions for the steady state Euler equations is the following.

At subsonic points one condition have to be specified on account of the elliptic part. This is the compressible analog of the  $(\text{div}, \text{curl})$  system for the incompressible case. This condition can be  $\rho \mathbf{V} \cdot \mathbf{n}$  where  $\mathbf{n}$  is the outward normal to the boundary. A compatibility condition has to hold, that is, the integral of that quantity around the boundary has to vanish, similar to the incompressible case. At inflow boundary points  $d$  more conditions has to be specified and the natural choice is  $(H, s, \beta)$ , as these quantities are carried into the domain by the convection operator  $Q$ . At outflow boundary points no condition is required for the operators  $Q$ .

At supersonic points the system is purely hyperbolic and require at the inflow  $d + 2$  conditions, and no conditions at outflow. In that case it is natural to specify the quantities  $(\mathbf{V}, H, s)$ , rather than using  $\beta$  for one of them.

## 5 Conclusion

Canonical forms for inviscid flow problems have been derived. In these forms the different types of subsystems have been separated using a set of new variables, namely, the canonical variables. These forms are upper triangular operator forms in which the building blocks of the systems reside on the diagonal and the interaction of the different type of subsystems is represented by the off diagonal blocks. This decomposition of systems of partial differential equations allows a better insight into the structure of solutions which is essential in constructing numerical solutions.

The forms described here suggest new iterative solutions of the inviscid equations, both in two and three dimensions. These will be discussed elsewhere.

The different approximations used in fluid dynamics over the years are clearly seen in the canonical form. Assuming the flow is of constant total pressure  $P$  one obtain a  $(\text{div}, \text{curl})$  system, which in two dimensions is nothing but the well known Cauchy-Riemann equations. The three dimensional implementation of a  $(\text{div}, \text{curl})$  system was not popular since the overdetermined system needs to be discretized carefully. As the canonical form suggests, the introduction of  $\beta$  into that overdetermined system fixes the major numerical difficulties related to existence of solutions.

For the compressible case, by assuming that the total enthalpy  $H$  and entropy  $s$  are constant, one obtains a system which is equivalent to the full potential equation. This is the upper left block in the canonical form.

## References

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